Appendix to

Frictionless Inflation

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A Proofs of propositions and auxiliary results

This appendix contains the proofs of all the propositions in the main text as well as some auxiliary results used in those proofs. Appendix A.1 contains the proofs of results in the estimation block. Appendix A.2 contains the proofs of all the results associated with the smoothing problem, including the solution to the filtering problem.

A.1 Proof of results for the estimation block

Proof of Proposition 1. For any t it follows from (7) that $L_{i,j,t}$ depends only on the parameter $\lambda_{i,j}$ and the realizations of the shock $\nu_{i,j,t}$ which is drawn from a Uniform (0,1), hence, $L_{i,j,t}$ does not depend on $x_{i,j,0}$. Iterating (6) backwards and using $Z^\star_{i,j,0} = 0$ yields,

$$Z^\star_{i,j,t} = t\mu_{i,j} + \sum_{k=1}^{t} \varepsilon_{i,j,k} \tag{P1.1}$$

Therefore, for any t this depends only on the parameters $\mu_{i,j}$ and $\sigma_{\varepsilon,i,j}$. For any $t > \tau^1_{i,j}$, equation (5) implies that $Z_{i,j,t-1} = Z^\star_{i,j,\tau^k_{i,j}} - x_{i,j,0}$ where $\tau^k_{i,j}$ denotes the last time period where a price change occurred. Subtract $Z_{i,j,t-1}$ on both sides of (5) and use $Z_{i,j,t-1} = Z^\star_{i,j,\tau^k_{i,j}} - x_{i,j,0}$ to obtain,

$$\Delta p_{i,j,t} = (Z^\star_{i,j,t} - Z^\star_{i,j,\tau^k_{i,j}})\,(1 - d_{i,j,t}) \tag{P1.2}$$

Finally, given that $Z^\star_{i,j,t}$ is independent of $x_{i,j,0}$ for any t, it remains to be shown that for $t > \tau^1_{i,j}$ also $d_{i,j,t}$ is independent of $x_{i,j,0}$. To see this substitute $Z_{i,j,t-1} = Z^\star_{i,j,\tau^k_{i,j}} - x_{i,j,0}$ in (2) to obtain:

$$d_{i,j,t} = \mathbb{1}\{Z^\star_{i,j,\tau^k_{i,j}} - Z^\star_{i,j,t} \in (\underline{x}_{i,j}, \bar{x}_{i,j})\}(1 - L_{i,j,t}) + \mathbb{1}\{Z^\star_{i,j,t} = Z^\star_{i,j,\tau^k_{i,j}}\}L_{i,j,t} \tag{P1.3}$$

Given that $Z^\star_{i,j,t}$ and $L_{i,j,t}$ do not depend on $x_{i,j,0}$, this completes the proof. □

A.2 Proof of results for filtering and smoothing

This section contains the proofs of all the relevant results used to derive the smoothed probability density function of cumulated frictionless inflation at any point in time. In total this section contains seven results. The first one is an algebraic fact on the product of normal densities (lemma 1). That is followed by two results that are the solution to the filtering problem in state-space representation (5), (6) and (7) (lemmas 2 and 3). This is followed by three results that characterise the smoothed density presented in the main text (propositions 2, 3 and 4) and the proof of smoothed estimates for the pure Calvo model (corollary 1). In addition to the the notation introduced at the beginning of section 3.3, the function $o(\cdot) : \mathbb{R} \to \mathbb{R}$ is used to denote any function that is equal to zero almost everywhere in the real line.

A.2.1 Auxiliary fact on the product of normal distributions

Lemma 1 *Let $x, y, \mu_x, a, c \in \mathbb{R}$, $\sigma_x, \sigma_y \in \mathbb{R}_{>0}$ and $\mu_y = ax + c$. Then,*

$$\frac{1}{\sigma_y}\phi\left(\frac{y-\mu_y}{\sigma_y}\right) \times \frac{1}{\sigma_x}\phi\left(\frac{x-\mu_x}{\sigma_x}\right) = \frac{1}{\tilde{\sigma}_y}\phi\left(\frac{y-\tilde{\mu}_y}{\tilde{\sigma}_y}\right) \times \frac{1}{\tilde{\sigma}_x}\phi\left(\frac{x-\tilde{\mu}_x}{\tilde{\sigma}_x}\right) \tag{L1.1}$$

where,

$$\tilde{\mu}_x = \frac{\sigma_x^2 a(y-c) + \sigma_y^2 \mu_x}{\sigma_y^2 + a^2\sigma_x^2} \tag{L1.2}$$

$$\tilde{\sigma}_x = \frac{\sigma_y \sigma_x}{\sqrt{\sigma_y^2 + a^2\sigma_x^2}} \tag{L1.3}$$

$$\tilde{\mu}_y = a\mu_x + c \tag{L1.4}$$

$$\tilde{\sigma}_y = \sqrt{\sigma_y^2 + a^2\sigma_x^2} \tag{L1.5}$$

Proof of Lemma 1. Using the definition of the standard normal probability density function:

$$\begin{aligned}\frac{1}{\sigma_y}\phi\left(\frac{y-\mu_y}{\sigma_y}\right) \times \frac{1}{\sigma_x}\phi\left(\frac{x-\mu_x}{\sigma_x}\right) &= \frac{1}{2\pi\sigma_y\sigma_x}\exp\left\{-\frac{(y-\mu_y)^2}{2\sigma_y^2} - \frac{(x-\mu_x)^2}{2\sigma_x^2}\right\} \\ &= \frac{1}{2\pi\sigma_y\sigma_x}\exp\left\{-\frac{1}{2\sigma_y^2\sigma_x^2}\underbrace{\left[\sigma_x^2(y-\mu_y)^2 + \sigma_y^2(x-\mu_x)^2\right]}_{(*)}\right\}\end{aligned} \tag{L1.6}$$

Given that $\mu_y = ax + c$, rearrange terms and define $\tilde{\mu}_x \equiv (\sigma_y^2 + a^2\sigma_x^2)^{-1}(\sigma_x^2 a(y-c) + \sigma_y^2\mu_x)$ to obtain,

$$
\begin{aligned}
(*) &= \sigma_x^2(y-c)^2 + (\sigma_x^2 a^2 + \sigma_y^2)\left(x^2 - 2x\tilde{\mu}_x\right) + \sigma_y^2\mu_x^2 \\
&= \sigma_x^2(y-c)^2 + (\sigma_x^2 a^2 + \sigma_y^2)\left(x^2 - 2x\tilde{\mu}_x + \tilde{\mu}_x^2\right) + \sigma_y^2\mu_x^2 - (\sigma_x^2 a^2 + \sigma_y^2)\tilde{\mu}_x^2 \\
&= (\sigma_x^2 a^2 + \sigma_y^2)\left(x - \tilde{\mu}_x\right)^2 + \underbrace{\sigma_x^2(y-c)^2 + \sigma_y^2\mu_x^2 - (\sigma_x^2 a^2 + \sigma_y^2)\tilde{\mu}_x^2}_{(**)} \qquad \text{(L1.7)}
\end{aligned}
$$

Using the definition of $\tilde{\mu}_x$ and rearranging terms yields,

$$
\begin{aligned}
(**) &= \sigma_x^2\sigma_y^2(\sigma_x^2 a^2 + \sigma_y^2)^{-1}\left[(y-c)^2 - 2a\mu_x(y-c) + a^2\mu_x^2\right] \\
&= \sigma_x^2\sigma_y^2(\sigma_x^2 a^2 + \sigma_y^2)^{-1}(y - (\underbrace{a\mu_x + c}_{\equiv\tilde{\mu}_y}))^2 \qquad \text{(L1.8)}
\end{aligned}
$$

Combine (L1.8) and (L1.7) and plug back in (L1.6),

$$
\frac{1}{\sigma_y}\phi\left(\frac{y-\mu_y}{\sigma_y}\right) \times \frac{1}{\sigma_x}\phi\left(\frac{x-\mu_x}{\sigma_x}\right) = \frac{1}{2\pi\sigma_y\sigma_x}\exp\left\{-\frac{1}{2}\left(\frac{(x-\tilde{\mu}_x)^2}{(\sigma_x^2 a^2 + \sigma_y^2)^{-1}\sigma_x^2\sigma_y^2} + \frac{(y-\tilde{\mu}_y)^2}{(\sigma_x^2 a^2 + \sigma_y^2)}\right)\right\}
$$

Finally, define $\tilde{\sigma}_x \equiv (\sigma_y^2 + a^2\sigma_x^2)^{-\frac{1}{2}}(\sigma_y\sigma_x)$ and $\tilde{\sigma}_y \equiv (\sigma_y^2 + a^2\sigma_x^2)^{\frac{1}{2}}$ and rearrange to obtain,

$$
\frac{1}{\sigma_y}\phi\left(\frac{y-\mu_y}{\sigma_y}\right) \times \frac{1}{\sigma_x}\phi\left(\frac{x-\mu_x}{\sigma_x}\right) = \underbrace{\frac{1}{(2\pi)^{\frac{1}{2}}\tilde{\sigma}_y}\exp\left\{-\frac{(y-\tilde{\mu}_y)^2}{2\tilde{\sigma}_y^2}\right\}}_{\frac{1}{\tilde{\sigma}_y}\phi\left(\frac{y-\tilde{\mu}_y}{\tilde{\sigma}_y}\right)} \underbrace{\frac{1}{(2\pi)^{\frac{1}{2}}\tilde{\sigma}_x}\exp\left\{-\frac{(x-\tilde{\mu}_x)^2}{2\tilde{\sigma}_x^2}\right\}}_{\frac{1}{\tilde{\sigma}_x}\phi\left(\frac{x-\tilde{\mu}_x}{\tilde{\sigma}_x}\right)}
$$

This completes the proof. □

A.2.2 The filtered density: main result

Lemma 2 *Consider an arbitrary time period* $t \in \mathbb{Z}_{[\tau^0,T]}$.

If $\exists\, k \in \mathbb{Z}_{[0,K]}$ *such that* $t = \tau^k$,

$$f_{Z_t^\star \,|\, Z^t;\Theta}\left(z^\star | z^t;\theta\right) \;=\; \delta\left(z^\star - c^k\right) \tag{L2.1}$$

where $c^k = 0$ *for* $k = 0$ *or* $c^k = z_{\tau^k} + x_0$ *for* $k \in \mathbb{Z}_{[1,K]}$.

Otherwise, suppose $\nexists\, k \in \mathbb{Z}_{[0,K]}$ *such that* $t = \tau^k$. *In that case, let* τ^k *denote the last period where* (L2.1) *holds, that is, let* k *be such that* $\nexists j \in \mathbb{Z}_{[0,K]}$ *that satisfies* $\tau^k < \tau^j < t$. *For a given combination of* t *and* k, *define:* $b \equiv t - \tau^k$, $\underline{Z}^k \equiv \mathbb{1}\{k = 0\}x_0 + \mathbb{1}\{k \geqslant 1\}c^k - \bar{x}$, $\bar{Z}^k \equiv \mathbb{1}\{k = 0\}x_0 + \mathbb{1}\{k \geqslant 1\}c^k - \underline{x}$ *and* $\mathcal{I}^k \equiv (\underline{Z}^k, \bar{Z}^k)$. *Then, ignoring terms that are zero almost everywhere, it holds that:*

$$f_{Z_t^\star \,|\, Z^t;\Theta}\left(z^\star | z^t;\theta\right) \;\propto\; \frac{1}{\sigma_b}\phi\left(\frac{z^\star - \mu_b^k}{\sigma_b}\right)\; \beta_b^k(z^\star)\, \mathbb{1}\left\{z^\star \in \mathcal{I}^k\right\} \tag{L2.2}$$

where $\beta_b^k(\cdot)$ *is given recursively by,*

$$\beta_b^k\left(x\right) = \begin{cases} 1, \text{ if } b = 1 \\ \int_{\underline{Z}^k}^{\bar{Z}^k} \frac{1}{\tilde{\sigma}_{b-1}}\, \phi\left(\frac{y - \tilde{\mu}_{b-1}^k(x)}{\tilde{\sigma}_{b-1}}\right)\, \beta_{b-1}^k\left(y\right) dy, \text{ if } b > 1 \end{cases} \tag{L2.3}$$

and the means and standard deviations of the distributions are given by,

$$\mu_b^k = b\mu + c^k \quad \text{and} \quad \sigma_b = \sqrt{b}\,\sigma_\varepsilon \tag{L2.4}$$

$$\tilde{\mu}_b^k(x) = \frac{c^k + bx}{b+1} \quad \text{and} \quad \tilde{\sigma}_b = \sqrt{\frac{b}{b+1}}\,\sigma_\varepsilon \tag{L2.5}$$

Proof of Lemma 2. Consider each of the cases separately.

Proof of (L2.1): If $t = \tau^0$, the distribution of $Z_t^\star$ must be degenerate at 0 since, by definition, $Z_{\tau^0}^\star = p_t^\star - p_{\tau^0}^\star$. Otherwise, if $t = \tau^k$ for some $k \in \mathbb{Z}_{[1,K]}$ it means that the time period considered corresponds to a period where a non-zero price change is observed. In that case, from (5) it must be the case that $d_t = 0$ and, hence, the distribution of $Z_t^\star$ must be degenerate at $z_t + x_0$. Defining $c^k = 0$ for $k = 0$ and $c^k = z_{\tau^k} + x_0$ for $k \in \mathbb{Z}_{[1,K]}$ and using the Dirac delta to denote a degenerate pdf, (L2.1) follows.

Proof of (L2.2): To establish (L2.2), start from Bayes' rule:

$$
\begin{aligned}
f_{Z_t^\star|Z^t;\Theta}(z^\star|z^t;\theta) &= f_{Z_t^\star|Z_t;Z^{t-1};\Theta}(z^\star|z_t;z^{t-1};\theta) \\
&= \frac{f_{Z_t|Z_t^\star;Z^{t-1};\Theta}(z_t|z^\star;z^{t-1};\theta) f_{Z_t^\star|Z^{t-1};\Theta}(z^\star|z^{t-1};\theta)}{\displaystyle\int f_{Z_t|Z_t^\star;Z^{t-1};\Theta}(z_t|a;z^{t-1};\theta)\, f_{Z_t^\star|Z^{t-1};\Theta}(a|z^{t-1};\theta)\, da}
\end{aligned} \tag{L2.6}
$$

Consider the first term in the numerator of (L2.6) and use the law of total probability:

$$
\begin{aligned}
f_{Z_t|Z_t^\star;Z^{t-1};\Theta}(z_t|z^\star;z^{t-1};\theta) &= f_{Z_t|Z_t^\star;Z^{t-1};L_t;\Theta}(z_t|z^\star;z^{t-1};0;\theta) \times f_{L_t|Z_t^\star;Z^{t-1};\Theta}(0|z^\star;z^{t-1};\theta) \\
&\quad + f_{Z_t|Z_t^\star;Z^{t-1};L_t;\Theta}(z_t|z^\star;z^{t-1};1;\theta) \times f_{L_t|Z_t^\star;Z^{t-1};\Theta}(1|z^\star;z^{t-1};\theta)
\end{aligned} \tag{L2.7}
$$

Since $\nexists k \in \mathbb{Z}_{[0,K]}$ such that $t = \tau^k$, it must be that this is a period of inaction and, hence, $z_t = z_{t-1}$ and $d_t = 1$. Moreover, from $\nexists j \in \mathbb{Z}_{[0,K]}$ such that $\tau^k < \tau^j < t$ it must also hold that $z_{t-1} = z_{\tau^k}$. Using the definition of d_t in (2) and the transition equation for the arrival of costless adjustment opportunities in (7), expression (L2.7) can be written as:

$$
f_{Z_t|Z_t^\star;Z^{t-1};\Theta}(z_t|z^\star;z^{t-1};\theta) = \mathbb{1}\{z^\star \in \mathcal{I}^k\} \times (1-\lambda) + \mathbb{1}\{z^\star = z_{\tau^k} + x_0\} \times \lambda \tag{L2.8}
$$

where $\mathcal{I}^k \equiv (\underline{Z}^k, \bar{Z}^k)$ and $\underline{Z}^k \equiv \mathbb{1}\{k=0\}x_0 + \mathbb{1}\{k \geqslant 1\}c^k - \bar{x}$ and $\bar{Z}^k \equiv \mathbb{1}\{k=0\}x_0 + \mathbb{1}\{k \geqslant 1\}c^k - \underline{x}$. Substituting (L2.8) in (L2.6) and re-arranging yields:

$$
\begin{aligned}
f_{Z_t^\star|Z^t;\Theta}(z^\star|z^t;\theta) &= \frac{f_{Z_t^\star|Z^{t-1};\Theta}(z^\star|z^{t-1};\theta)}{C_t}\mathbb{1}\{z^\star \in \mathcal{I}^k\} \\
&\quad + \underbrace{\frac{\lambda}{(1-\lambda)}\frac{f_{Z_t^\star|Z^{t-1};\Theta}(z^\star|z^{t-1};\theta)}{C_t}\mathbb{1}\{z^\star = z_{\tau^k} + x_0\}}_{=o(z^\star)}
\end{aligned} \tag{L2.9}
$$

where $C_t \equiv \displaystyle\int_{\underline{Z}^k}^{\bar{Z}^k} f_{Z_t^\star|Z^{t-1};\Theta}(a|z^{t-1};\theta)\, da$ and the second term on the RHS is zero almost everywhere in the real line. Considering now the second term in the numerator of (L2.6) and use the Chapman-Kolmogorov equation to obtain:

$$
f_{Z_t^\star|Z^{t-1};\Theta}(z^\star|z^{t-1};\theta) = \int f_{Z_t^\star|Z_{t-1}^\star;\Theta}(z^\star|\tilde{z}^\star;\theta) f_{Z_{t-1}^\star|Z^{t-1};\Theta}(\tilde{z}^\star|z^{t-1};\theta)\, d\tilde{z}^\star \tag{L2.10}
$$

Using the transition equation for the cumulated frictionless inflation in (6) and the normality of the idiosyncratic shocks (ε_t) yields that,

$$f_{Z_t^\star|Z_{t-1}^\star;\Theta}(z^\star|\tilde{z}^\star;\theta) = \frac{1}{\sigma_\varepsilon}\phi\left(\frac{z^\star - (\mu + \tilde{z}^\star)}{\sigma_\varepsilon}\right) \tag{L2.11}$$

Substituting (L2.10) and (L2.11) in (L2.9),

$$f_{Z_t^\star|Z^t;\Theta}(z^\star|z^t;\theta) = C_t^{-1}\int \frac{1}{\sigma_\varepsilon}\phi\left(\frac{z^\star - (\mu + \tilde{z}^\star)}{\sigma_\varepsilon}\right) f_{Z_{t-1}^\star|Z^{t-1};\Theta}(\tilde{z}^\star|z^{t-1};\theta)\, d\tilde{z}^\star\, \mathbb{1}\{z^\star \in \mathcal{I}^k\} + o(z^\star) \tag{L2.12}$$

Equation (L2.12) is a *filtering forward recursion*. It expresses the filtered pdf a given time period as a function of the filtered pdf in the previous time period. Filtering forward recursions are common in the nonlinear non-Gaussian filtering literature (see, for instance, Kitagawa (1987, equation 2.3) or Särkkä (2013, theorem 4.1)). The key difference of (L2.12) is that it holds only for *inaction periods* (i.e. $\nexists k \in \mathbb{Z}_{[0,K]}$ such that $t = \tau^k$) for which the last period where $Z^\star$ is known is τ^k (i.e. $\nexists j \in \mathbb{Z}_{[0,K]}$ that satisfies $\tau^k < \tau^j < t$). To complete the proof it remains to be shown that (L2.2) satisfies (L2.12) for any such time period. This is shown by induction.

Base case: Suppose t is an inaction period and that $t = \tau^k + 1$ (so that $b = 1$). In that case, using that at $t = \tau^k$ the filtered pdf is given by (L2.1), then (L2.12) reads,

$$\begin{aligned}
f_{Z_{\tau^k+1}^\star|Z^{\tau^k+1};\Theta}(z^\star|z^{\tau^k+1};\theta) &= C_{\tau^k+1}^{-1}\int \frac{1}{\sigma_\varepsilon}\phi\left(\frac{z^\star - (\mu + \tilde{z}^\star)}{\sigma_\varepsilon}\right) \delta(\tilde{z}^\star - c^k)\, d\tilde{z}^\star\, \mathbb{1}\{z^\star \in \mathcal{I}^k\} + o(z^\star) \\
&= C_{\tau^k+1}^{-1}\frac{1}{\sigma_\varepsilon}\phi\left(\frac{z^\star - (\mu + c^k)}{\sigma_\varepsilon}\right)\, \mathbb{1}\{z^\star \in \mathcal{I}^k\} + o(z^\star) \\
&= C_{\tau^k+1}^{-1}\frac{1}{\sigma_1}\phi\left(\frac{z^\star - \mu_1^k}{\sigma_1}\right) \beta_1^k(z^\star)\, \mathbb{1}\{z^\star \in \mathcal{I}^k\} + o(z^\star) \qquad \text{(L2.13)}
\end{aligned}$$

where: (i) the second equality uses the properties of the Dirac delta function and (ii) the third equality uses the definitions of μ_b^k and σ_b in (L2.4) with $b = 1$ and the fact that $\beta_b^k(x) = 1$ for $b = 1$ from (L2.3). Therefore, (L2.2) satisfies (L2.12) for the base case.

Induction step: Suppose $t-1$ and t are both inaction periods and they are such that $\nexists j \in \mathbb{Z}_{[0,K]}$ that satisfies $\tau^k < \tau^j < t-1$. Define $b \equiv t - \tau^k$ and suppose (L2.2) holds for $t-1$, that is, the filtered pdf at $t-1$ can be written as:

$$f_{Z_{t-1}^\star|Z^{t-1};\Theta}(\tilde{z}^\star|z^{t-1};\theta) = C_{t-1}\frac{1}{\sigma_{b-1}}\phi\left(\frac{\tilde{z}^\star - \mu_{b-1}^k}{\sigma_{b-1}}\right) \beta_{b-1}^k(\tilde{z}^\star)\, \mathbb{1}\{\tilde{z}^\star \in \mathcal{I}^k\} + o(\tilde{z}^\star) \tag{L2.14}$$

where C_{t-1} is a normalisation constant. Substituting (L2.14) in (L2.12) yields,

$$f_{Z_t^\star|Z^t;\Theta}(z^\star|z^t;\theta) = \frac{C_{t-1}}{C_t}\int_{\underline{Z}^k}^{\bar{Z}^k} \frac{1}{\sigma_\varepsilon}\phi\left(\frac{z^\star-(\mu+\tilde{z}^\star)}{\sigma_\varepsilon}\right)\frac{1}{\sigma_{b-1}}\phi\left(\frac{\tilde{z}^\star-\mu_{b-1}^k}{\sigma_{b-1}}\right)\beta_{b-1}^k(\tilde{z}^\star)d\tilde{z}^\star\mathbb{1}\{z^\star\in\mathcal{I}^k\}+o(z^\star)$$

$$= \frac{C_{t-1}}{C_t}\frac{1}{\sigma_b}\phi\left(\frac{z^\star-\mu_b^k}{\sigma_b}\right)\int_{\underline{Z}^k}^{\bar{Z}^k}\frac{1}{\tilde{\sigma}_{b-1}}\phi\left(\frac{\tilde{z}^\star-\tilde{\mu}_{b-1}^k(z^\star)}{\tilde{\sigma}_{b-1}}\right)\beta_{b-1}^k(\tilde{z}^\star)d\tilde{z}^\star\mathbb{1}\{z^\star\in\mathcal{I}^k\}+o(z^\star)$$

$$= \frac{C_{t-1}}{C_t}\frac{1}{\sigma_b}\phi\left(\frac{z^\star-\mu_b^k}{\sigma_b}\right)\beta_b^k(z^\star)\mathbb{1}\{z^\star\in\mathcal{I}^k\}+o(z^\star) \tag{L2.15}$$

where: (i) the first equality follows from combining (L2.14) in (L2.12) and rearranging terms; (ii) the second equality uses lemma 1 to combine the two normal pdfs in the integral and the definitions in (L2.4) and (L2.5); (iii) the third equality uses the definition of (L2.3).[46] Therefore, the filtered pdf is again given by (L2.2).

This completes the proof. □

A.2.3 The filtered density: auxiliary result

Lemma 3 *Consider an arbitrary time period $t \in \mathbb{Z}_{[\tau^0,T]}$. If $t=\tau^0$ then,*

$$f_{Z_t^\star\,|\,\Theta}\left(z^\star|\Theta\right) \;=\; \delta\left(z^\star\right) \tag{L3.1}$$

Otherwise, for any $t>\tau^0$ let k be such that $\nexists j\in\mathbb{Z}_{[0,K]}:\tau^k<\tau^j<t$. Define $b\equiv t-\tau^k$, $\underline{Z}^k\equiv\mathbb{1}\{k=0\}x_0+\mathbb{1}\{k\geqslant 1\}c^k-\bar{x}$ and $\bar{Z}^k\equiv\mathbb{1}\{k=0\}x_0+\mathbb{1}\{k\geqslant 1\}c^k-\underline{x}$. Then,

$$f_{Z_t^\star\,|\,Z^{t-1};\Theta}\left(z^\star|z^{t-1};\Theta\right) \;\propto\; \frac{1}{\sigma_b}\phi\left(\frac{z^\star-\mu_b^k}{\sigma_b}\right)\;\beta_b^k(z^\star) \tag{L3.2}$$

where $\beta_b^k(\cdot)$ is given by (L2.3)*, μ_b^k and σ_b are given by* (L2.4)*.*

Proof of Lemma 3. For the initial time period $t=\tau^0$ expression (L3.1) follows from the fact that the distribution is degenerate at zero since, by definition, $Z_t^\star=p_t^\star-p_{\tau^0}^\star$. For any other time period, use the Chapman-Kolmogorov equation and the pdf for $Z_t^\star$ conditional on $Z_{t-1}^\star$:

$$f_{Z_t^\star|Z^{t-1};\Theta}(z^\star|z^{t-1};\theta) = \int\frac{1}{\sigma_\varepsilon}\phi\left(\frac{z^\star-(\mu+\tilde{z}^\star)}{\sigma_\varepsilon}\right)f_{Z_{t-1}^\star|Z^{t-1};\Theta}(\tilde{z}^\star|z^{t-1};\theta)\,d\tilde{z}^\star \tag{L3.3}$$

It remains to be shown that for any t, the expression for the pdf in (L3.2) satisfies (L3.3). According to the expressions for the filtered pdf in lemma 2 there are two cases to be verified.

[46] In the second equality Lemma 1 is invoked with: $y=z^\star$, $x=\tilde{z}^\star$, $a=1$, $c=\mu$, $\sigma_y=\sigma_\varepsilon$, $\sigma_x=\sigma_{b-1}$ and $\mu_x=\mu_{b-1}^k$.

First, consider the case where $t-1$ is either a period for which a non-zero price change is observed or the initial period, that is, $\exists k \in \mathbb{Z}_{[0,K]}$ such that $t-1=\tau^k$. Combining the filtered density in (L2.1) with (L3.3) yields,

$$
\begin{aligned}
f_{Z_t^\star|Z^{t-1};\Theta}(z^\star|z^{t-1};\theta) &= \int \frac{1}{\sigma_\varepsilon}\phi\left(\frac{z^\star-(\mu+\tilde{z}^\star)}{\sigma_\varepsilon}\right)\delta(\tilde{z}^\star-c^k)\,d\tilde{z}^\star \\
&= \frac{1}{\sigma_\varepsilon}\phi\left(\frac{z^\star-(\mu+c^k)}{\sigma_\varepsilon}\right) \\
&= \frac{1}{\sigma_1}\phi\left(\frac{z^\star-\mu_1^k}{\sigma_1}\right)\beta_1^k(z^\star) \qquad \text{(L3.4)}
\end{aligned}
$$

where: (i) the second equality uses the properties of the Dirac delta function; (ii) the third equality used the definitions of μ_b^k and σ_b in (L2.4) for $b=1$ and that $\beta_b^k(x)=1\ \forall x$ if $b=1$ from (L2.3). Therefore, (L3.2) holds.

Second, consider the case where $t-1$ is a period of inaction and the last period for which the value of $Z^\star$ is known is given by τ^k. In that case, the filtered density is given by (L2.2) and (L3.3) reads,

$$
\begin{aligned}
f_{Z_t^\star|Z^{t-1};\Theta}(z^\star|z^{t-1};\theta) &= \int_{\underline{Z}^k}^{\bar{Z}^k} \frac{1}{\sigma_\varepsilon}\phi\left(\frac{z^\star-(\mu+\tilde{z}^\star)}{\sigma_\varepsilon}\right)\frac{1}{\sigma_{b-1}}\phi\left(\frac{\tilde{z}^\star-\mu_{b-1}^k}{\sigma_{b-1}}\right)\beta_{b-1}^k(\tilde{z}^\star)\,d\tilde{z}^\star \\
&= C_t\,\frac{1}{\sigma_b}\phi\left(\frac{z^\star-\mu_b^k}{\sigma_b}\right)\int_{\underline{Z}^k}^{\bar{Z}^k}\frac{1}{\tilde{\sigma}_{b-1}}\phi\left(\frac{\tilde{z}^\star-\tilde{\mu}_{b-1}^k(z^\star)}{\tilde{\sigma}_{b-1}}\right)\beta_{b-1}^k(\tilde{z}^\star)d\tilde{z}^\star \\
&= C_t\,\frac{1}{\sigma_b}\phi\left(\frac{z^\star-\mu_b^k}{\sigma_b}\right)\beta_b^k(z^\star) \qquad \text{(L3.5)}
\end{aligned}
$$

where: (i) follows from substituting (L2.2) in (L3.3) and re-arranging; (ii) the second equality from applying lemma 1 to combine the two normal densities (similarly to the derivation of (L2.15)); (iii) the third equality follows by using the definition of $\beta_b^k(z^\star)$ in (L2.3). This completes the proof. □

A.2.4 Proof of Proposition 2

Proof of proposition 2. For the period $t=\tau^0$, $Z^\star$ is degenerate at zero since, by definition, $Z_t^\star=p_t^\star-p_{\tau^0}^\star$. For any other period where a non-zero price change is observed, $t=\tau^k$ for some $k\in\mathbb{Z}_{[1,K]}$, $Z^\star$ is degenerate at the value that closes the price gap, that is, $Z_t^\star=z_{\tau^k}+x_0$ (see measurement equation (5)). This completes the proof. □

A.2.5 Proof of Proposition 3

Proof of proposition 3. This proof has two blocks. First, a smoothing backward recursion that holds for any $t \in (\tau^k, \tau^{k+1})$ and some $k \in \mathbb{Z}_{[0,K-1]}$ is derived. Second, we verify by induction that (12) satisfies that recursion for any $t \in (\tau^k, \tau^{k+1})$. To derive the smoothing backward recursion we start from,

$$
\begin{aligned}
f_{Z_t^\star|Z^T;\Theta}(z^\star|z^T;\theta) &= \int f_{Z_t^\star, Z_{t+1}^\star|Z^T;\Theta}(z^\star, \tilde{z}^\star|z^T;\theta)\, d\tilde{z}^\star \\
&= \int f_{Z_{t+1}^\star|Z^T;\Theta}(\tilde{z}^\star|z^T;\theta)\, f_{Z_t^\star|Z_{t+1}^\star;Z^T;\Theta}(z^\star|\tilde{z}^\star;z^T;\theta)\, d\tilde{z}^\star \\
&= \int f_{Z_{t+1}^\star|Z^T;\Theta}(\tilde{z}^\star|z^T;\theta)\, f_{Z_t^\star|Z_{t+1}^\star;Z^t;\Theta}(z^\star|\tilde{z}^\star;z^t;\theta)\, d\tilde{z}^\star \\
&= \int \frac{f_{Z_{t+1}^\star|Z^T;\Theta}(\tilde{z}^\star|z^T;\theta)\, f_{Z_{t+1}^\star|Z_t^\star;\Theta}(\tilde{z}^\star|z^\star;\theta)\, f_{Z_t^\star|Z^t;\Theta}(z^\star|z^t;\theta)}{f_{Z_{t+1}^\star|Z^t;\Theta}(\tilde{z}^\star|z^t;\theta)}\, d\tilde{z}^\star \\
&= f_{Z_t^\star|Z^t;\Theta}(z^\star|z^t;\theta) \int \frac{f_{Z_{t+1}^\star|Z_t^\star;\Theta}(\tilde{z}^\star|z^\star;\theta)\, f_{Z_{t+1}^\star|Z^T;\Theta}(\tilde{z}^\star|z^T;\theta)}{f_{Z_{t+1}^\star|Z^t;\Theta}(\tilde{z}^\star|z^t;\theta)}\, d\tilde{z}^\star \qquad \text{(P3.1)}
\end{aligned}
$$

This derivation is similar to Kitagawa (1987, equation 2.4) or Särkkä (2013, theorem 8.1). For any $t \in (\tau^k, \tau^{k+1})$, the filtered pdf $f_{Z_t^\star|Z^t;\Theta}(z^\star|z^t;\theta)$ is given by (L2.2) whereas the term in the denominator of the expression inside the integral is given by (L3.2). Let $b = t - \tau^k$, substitute using (L2.2), (L2.11) and (L3.2) and re-arrange to obtain,

$$
\begin{aligned}
f_{Z_t^\star|Z^T;\Theta}(z^\star|z^T;\theta) = \frac{C_{t-1}}{C_t^2}\frac{1}{\sigma_b}\phi\left(\frac{z^\star-\mu_b^k}{\sigma_b}\right)\beta_b^k(z^\star)\mathbb{1}\{z^\star \in \mathcal{I}^k\} \int \frac{\frac{1}{\sigma_\varepsilon}\phi\left(\frac{\tilde{z}^\star-(\mu+z^\star)}{\sigma_\varepsilon}\right) f_{Z_{t+1}^\star|Z^T;\Theta}(\tilde{z}^\star|z^T;\theta)}{\frac{1}{\sigma_{b+1}}\phi\left(\frac{\tilde{z}^\star-\mu_{b+1}^k}{\sigma_{b+1}}\right)\beta_{b+1}^k(\tilde{z}^\star)}\, d\tilde{z}^\star \\
+ \underbrace{o(z^\star)\frac{1}{C_t}\int \frac{\frac{1}{\sigma_\varepsilon}\phi\left(\frac{\tilde{z}^\star-(\mu+z^\star)}{\sigma_\varepsilon}\right) f_{Z_{t+1}^\star|Z^T;\Theta}(\tilde{z}^\star|z^T;\theta)}{\frac{1}{\sigma_{b+1}}\phi\left(\frac{\tilde{z}^\star-\mu_{b+1}^k}{\sigma_{b+1}}\right)\beta_{b+1}^k(\tilde{z}^\star)}\, d\tilde{z}^\star}_{=o(z^\star)} \qquad \text{(P3.2)}
\end{aligned}
$$

Let $\tilde{C}_t \equiv C_{t-1}/C_t^2$ and re-arrange terms to obtain,

$$
f_{Z_t^\star|Z^T;\Theta}(z^\star|z^T;\theta) = \tilde{C}_t\beta_b^k(z^\star)\mathbb{1}\{z^\star \in \mathcal{I}^k\}\int \frac{\frac{1}{\sigma_b}\phi\left(\frac{z^\star-\mu_b^k}{\sigma_b}\right)\frac{1}{\sigma_\varepsilon}\phi\left(\frac{\tilde{z}^\star-(\mu+z^\star)}{\sigma_\varepsilon}\right) f_{Z_{t+1}^\star|Z^T;\Theta}(\tilde{z}^\star|z^T;\theta)}{\frac{1}{\sigma_{b+1}}\phi\left(\frac{\tilde{z}^\star-\mu_{b+1}^k}{\sigma_{b+1}}\right)\beta_{b+1}^k(\tilde{z}^\star)}\, d\tilde{z}^\star + o(z^\star) \qquad \text{(P3.3)}
$$

Equation (P3.3) is a *smoothing backward recursion* as it relates the smoothed pdf at time t to the smoothed pdf at time $t+1$. Whilst (P3.1) holds in general for any Markov state-space representation, (P3.3) is specific to the state-space representation in (5) to (7) as it uses the expressions for the filtered density and its auxiliary form derived in lemmas 2 and 3 and that hold for any $t \in (\tau^k, \tau^{k+1})$ for some $k \in \mathbb{Z}_{[0,K-1]}$. It remains to be shown that expression (12) satisfies (P3.3) for any $t \in (\tau^k, \tau^{k+1})$ for some $k \in \mathbb{Z}_{[0,K-1]}$. To do so, the proof proceeds by induction.

Base case: Let $t = \tau^{k+1} - 1$ so that $b = \Delta^k - 1$. In that case, the smoothed pdf at $t+1$ is given by (11). Substituting that in (P3.3),

$$
\begin{aligned}
f_{Z^\star_{\tau^{k+1}-1}|Z^T;\Theta}(z^\star|z^T;\theta) &= \tilde{C}_{\tau^{k+1}-1}\beta^k_{\Delta^k-1}(z^\star)\mathbb{1}\{z^\star \in \mathcal{I}^k\} \\
&\quad \times \int \frac{\frac{1}{\sigma_{\Delta^k-1}}\phi\left(\frac{z^\star-\mu^k_{\Delta^k-1}}{\sigma_{\Delta^k-1}}\right)\frac{1}{\sigma_\varepsilon}\phi\left(\frac{\tilde{z}^\star-(\mu+z^\star)}{\sigma_\varepsilon}\right)\delta(\tilde{z}^\star - c^{k+1})}{\frac{1}{\sigma_{\Delta^k}}\phi\left(\frac{\tilde{z}^\star-\mu^k_{\Delta^k}}{\sigma_{\Delta^k}}\right)\beta^k_{\Delta^k}(\tilde{z}^\star)} d\tilde{z}^\star + o(z^\star) \\
&= \tilde{C}_{\tau^{k+1}-1}\,\beta^k_{\Delta^k-1}(z^\star)\,\mathbb{1}\{z^\star \in \mathcal{I}^k\}\frac{\frac{1}{\sigma_{\Delta^k-1}}\phi\left(\frac{z^\star-\mu^k_{\Delta^k-1}}{\sigma_{\Delta^k-1}}\right)\frac{1}{\sigma_\varepsilon}\phi\left(\frac{c^{k+1}-(\mu+z^\star)}{\sigma_\varepsilon}\right)}{\frac{1}{\sigma_{\Delta^k}}\phi\left(\frac{c^{k+1}-\mu^k_{\Delta^k}}{\sigma_{\Delta^k}}\right)\beta^k_{\Delta^k}(c^{k+1})} + o(z^\star) \\
&= \check{C}_{\tau^{k+1}-1}\,\beta^k_{\Delta^k-1}(z^\star)\,\mathbb{1}\{z^\star \in \mathcal{I}^k\}\frac{\frac{1}{\check{\sigma}^k_{\Delta^k-1}}\phi\left(\frac{z^\star-\check{\mu}^k_{\Delta^k-1}}{\check{\sigma}^k_{\Delta^k-1}}\right)\phi\left(\frac{c^{k+1}-\mu^k_{\Delta^k}}{\sigma_{\Delta^k}}\right)}{\phi\left(\frac{c^{k+1}-\mu^k_{\Delta^k}}{\sigma_{\Delta^k}}\right)} + o(z^\star) \\
&= \check{C}_{\tau^{k+1}-1}\,\frac{1}{\check{\sigma}^k_{\Delta^k-1}}\phi\left(\frac{z^\star-\check{\mu}^k_{\Delta^k-1}}{\check{\sigma}^k_{\Delta^k-1}}\right)\,\beta^k_{\Delta^k-1}(z^\star)\,\chi^k_{\Delta^k-1}(z^\star)\,\mathbb{1}\{z^\star \in \mathcal{I}^k\} + o(z^\star)
\end{aligned}
\tag{P3.4}
$$

where: (i) the second equality follows from the properties of the Dirac delta function; (ii) the third equality uses $\check{C}_{\tau^{k+1}-1} \equiv \tilde{C}_{\tau^{k+1}-1}/\beta^k_{\Delta^k}(c^{k+1})$ and lemma 1 to combine the two normal densities in the numerator as well as the definitions of $\check{\mu}^k_b$ and $\check{\sigma}^k_b$ in (15); (iii) the fourth equality simply rearranges terms and uses that $\chi^k_{\Delta^k-1}(x) = 1, \forall x$ from the recursive definition in (14).[47] This shows that (12) satisfies the smoothing backward recursion (P3.3) for $t = \tau^{k+1} - 1$.

[47] In the third equality lemma 1 is invoked with: $y = c^{k+1}$, $x = z^\star$, $a = 1$, $c = \mu$, $\sigma_y = \sigma_\varepsilon$, $\sigma_x = \sigma_{\Delta^k-1}$ and $\mu_x = \mu^k_{\Delta^k-1}$.

Induction step: Consider two time periods $t, t+1 \in (\tau^k, \tau^{k+1})$ and define $b = t - \tau^k$. Suppose that (12) holds for $t+1$ and substitute that in (P3.3) to obtain,

$$f_{Z_t^\star|Z^T;\Theta}(z^\star|z^T;\theta) = \tilde{C}_t \check{C}_{t+1} \beta_b^k(z^\star) \mathbb{1}\{z^\star \in \mathcal{I}^k\}$$
$$\times \underbrace{\int_{\underline{Z}^k}^{\bar{Z}^k} \frac{\frac{1}{\sigma_b}\phi\left(\frac{z^\star-\mu_b^k}{\sigma_b}\right)\frac{1}{\sigma_\varepsilon}\phi\left(\frac{\tilde{z}^\star-(\mu+z^\star)}{\sigma_\varepsilon}\right)\frac{1}{\check{\sigma}_{b+1}^k}\phi\left(\frac{\tilde{z}^\star-\check{\mu}_{b+1}^k}{\check{\sigma}_{b+1}^k}\right)\beta_{b+1}^k(\tilde{z}^\star)\chi_{b+1}^k(\tilde{z}^\star)}{\frac{1}{\sigma_{b+1}}\phi\left(\frac{\tilde{z}^\star-\mu_{b+1}^k}{\sigma_{b+1}}\right)\beta_{b+1}^k(\tilde{z}^\star)} d\tilde{z}^\star}_{=(*)} + o(z^\star) \tag{P3.5}$$

where $\check{C}_{t+1}$ denotes the normalisation constant of the smoothed pdf at $t+1$. Looking at the integral term only, we have that:

$$\begin{aligned}
(*) &= \int_{\underline{Z}^k}^{\bar{Z}^k} \frac{\frac{1}{\sigma_{b+1}}\phi\left(\frac{\tilde{z}^\star-\mu_{b+1}^k}{\sigma_{b+1}}\right)\frac{1}{\tilde{\sigma}_b}\phi\left(\frac{z^\star-\tilde{\mu}_b^k(\tilde{z}^\star)}{\tilde{\sigma}_b}\right)\frac{1}{\check{\sigma}_{b+1}^k}\phi\left(\frac{\tilde{z}^\star-\check{\mu}_{b+1}^k}{\check{\sigma}_{b+1}^k}\right)\chi_{b+1}^k(\tilde{z}^\star)}{\frac{1}{\sigma_{b+1}}\phi\left(\frac{\tilde{z}^\star-\mu_{b+1}^k}{\sigma_{b+1}}\right)} d\tilde{z}^\star \\
&= \int_{\underline{Z}^k}^{\bar{Z}^k} \frac{1}{\check{\sigma}_b^k}\phi\left(\frac{z^\star-\check{\mu}_b^k}{\check{\sigma}_b^k}\right)\frac{1}{\ddot{\sigma}_{b+1}^k}\phi\left(\frac{\tilde{z}^\star-\ddot{\mu}_{b+1}^k(z^\star)}{\ddot{\sigma}_{b+1}^k}\right)\chi_{b+1}^k(\tilde{z}^\star)\, d\tilde{z}^\star \\
&= \frac{1}{\check{\sigma}_b^k}\phi\left(\frac{z^\star-\check{\mu}_b^k}{\check{\sigma}_b^k}\right)\int_{\underline{Z}^k}^{\bar{Z}^k} \frac{1}{\ddot{\sigma}_{b+1}^k}\phi\left(\frac{\tilde{z}^\star-\ddot{\mu}_{b+1}^k(z^\star)}{\ddot{\sigma}_{b+1}^k}\right)\chi_{b+1}^k(\tilde{z}^\star)\, d\tilde{z}^\star \\
&= \frac{1}{\check{\sigma}_b^k}\phi\left(\frac{z^\star-\check{\mu}_b^k}{\check{\sigma}_b^k}\right)\chi_b^k(z^\star)
\end{aligned} \tag{P3.6}$$

where: (i) the first equality follows from cancelling out the terms $\beta_{b+1}^k(\tilde{z}^\star)$ and by using lemma 1 to combine the first two normal densities in the numerator along with the definitions of $\tilde{\mu}_b^k(x)$ and $\tilde{\sigma}_b$ in (16); (ii) the second equality follows from cancelling out the first density in the numerator with the one in the denominator and using again lemma 1 to combine the two remaining pdfs along with the definitions of $\ddot{\mu}_b^k$ and $\ddot{\sigma}_b^k$ in (17); (iii) the third equality takes out of the integral terms that do not depend on $\tilde{z}^\star$; (iv) the fourth equality follows from using the definition of $\chi_b^k(\cdot)$ in (14).[48]

[48] In the first equality, lemma 1 is used with: $y = \tilde{z}^\star$, $x = z^\star$, $a = 1$, $c = \mu$, $\sigma_x = \sigma_b$, $\sigma_y = \sigma_\varepsilon$ and $\mu_x = \mu_b^k$. In the second equality lemma 1 is again used but now with: $y = z^\star$, $x = \tilde{z}^\star$, $a = b/(b+1)$, $c = c^k/(b+1)$, $\sigma_y = \tilde{\sigma}_b$, $\sigma_x = \check{\sigma}_{b+1}^k$ and $\mu_x = \check{\mu}_{b+1}^k$.

Finally, define $\check{C}_t = \tilde{C}_t \check{C}_{t+1}$ and plug back (P3.6) into (P3.5) to obtain,

$$f_{Z_t^\star|Z^T;\Theta}(z^\star|z^T;\theta) = \check{C}_t \, \frac{1}{\check{\sigma}_b^k}\phi\left(\frac{z^\star - \check{\mu}_b^k}{\check{\sigma}_b^k}\right) \, \beta_b^k(z^\star) \, \chi_b^k(z^\star) \, \mathbb{1}\{z^\star \in \mathcal{I}^k\} + o(z^\star) \tag{P3.7}$$

Therefore, (12) is again the solution for the smoothing backward recursion in (P3.3).

This completes the proof. □

A.2.6 Proof of Proposition 4

Proof of proposition 4. This proof is similar to that of proposition 3. We will show that (18) is the smoothed density for any $t \in (\tau^K, T]$. For the base case $t = T$, the smoothed pdf must be equal to the filtered pdf. Therefore, we start from the filtered pdf expression in (L2.2) with $k = K$, $t = T$ and $b = \Delta^K$ and show that it is equivalent to (18),

$$\begin{aligned} f_{Z_T^\star|Z^T;\Theta}(z^\star|z^T;\theta) &= C_T \, \frac{1}{\sigma_{\Delta^K}}\phi\left(\frac{z^\star - \mu_{\Delta^K}^K}{\sigma_{\Delta^K}}\right) \, \beta_{\Delta^K}^K(z^\star) \, \mathbb{1}\{z^\star \in \mathcal{I}^K\} + o(z^\star) \\ &= C_T \, \frac{1}{\sigma_{\Delta^K}}\phi\left(\frac{z^\star - \mu_{\Delta^K}^K}{\sigma_{\Delta^K}}\right) \, \beta_{\Delta^K}^K(z^\star) \, \iota_{\Delta^K}^K(z^\star) \, \mathbb{1}\{z^\star \in \mathcal{I}^K\} + o(z^\star) \end{aligned} \tag{P4.1}$$

where C_T is the integration constant for the filtered pdf in (L2.2) for $t = T$ and the second equality follows from $\iota_{\Delta^K}^K(x) = 1 \ \forall x$ as defined in (14). Therefore, (18) is satisfied. For $t \in (\tau^K, T)$, it means that the last period where $Z^\star$ is known is τ^K and, in that case, the smoothing backward recursion in (P3.3) reads as:

$$f_{Z_t^\star|Z^T;\Theta}(z^\star|z^T;\theta) = \tilde{C}_t \beta_b^K(z^\star) \mathbb{1}\{z^\star \in \mathcal{I}^K\} \int \frac{\frac{1}{\sigma_b}\phi\left(\frac{z^\star - \mu_b^K}{\sigma_b}\right) \frac{1}{\sigma_\varepsilon}\phi\left(\frac{\tilde{z}^\star - (\mu + z^\star)}{\sigma_\varepsilon}\right) f_{Z_{t+1}^\star|Z^T;\Theta}(\tilde{z}^\star|z^T;\theta)}{\frac{1}{\sigma_{b+1}}\phi\left(\frac{\tilde{z}^\star - \mu_{b+1}^K}{\sigma_{b+1}}\right) \beta_{b+1}^K(\tilde{z}^\star)} d\tilde{z}^\star + o(z^\star) \tag{P4.2}$$

where $b = t - \tau^K$. Assuming (18) holds for $t + 1 \in (\tau^K, T]$, the induction step simply requires showing (18) solves (P4.2) . Substituting (18) in the smoothed pdf in (P4.2) yields,

$$\begin{aligned} f_{Z_t^\star|Z^T;\Theta}(z^\star|z^T;\theta) &= \tilde{C}_t \, \check{C}_{t+1} \beta_b^K(z^\star) \mathbb{1}\{z^\star \in \mathcal{I}^K\} \\ &\times \int_{\underline{Z}^K}^{\bar{Z}^K} \frac{\frac{1}{\sigma_b}\phi\left(\frac{z^\star - \mu_b^K}{\sigma_b}\right) \frac{1}{\sigma_\varepsilon}\phi\left(\frac{\tilde{z}^\star - (\mu + z^\star)}{\sigma_\varepsilon}\right) \frac{1}{\sigma_{b+1}}\phi\left(\frac{\tilde{z}^\star - \mu_{b+1}^K}{\sigma_{b+1}}\right) \beta_{b+1}^K(\tilde{z}^\star) \iota_{b+1}^K(\tilde{z}^\star)}{\frac{1}{\sigma_{b+1}}\phi\left(\frac{\tilde{z}^\star - \mu_{b+1}^K}{\sigma_{b+1}}\right) \beta_{b+1}^K(\tilde{z}^\star)} d\tilde{z}^\star + o(z^\star) \end{aligned} \tag{P4.3}$$

where $\check{C}_{t+1}$ is the normalisation constant of the smoothed pdf at $t + 1$. Define $\check{C}_t \equiv \tilde{C}_t \, \check{C}_{t+1}$, cancel out terms inside the integral, bring the first pdf outside the integral and use the definition of $\iota_b^K(x)$ in (19) to write (P4.3) as,

$$f_{Z_t^\star|Z^T;\Theta}(z^\star|z^T;\theta) = \check{C}_t \frac{1}{\sigma_b}\phi\left(\frac{z^\star - \mu_b^K}{\sigma_b}\right) \beta_b^K(z^\star)\, \iota_b^K(z^\star)\, \mathbb{1}\{z^\star \in \mathcal{I}^K\} + o(z^\star) \qquad \text{(P4.4)}$$

Therefore (18) solves the backward smoothing recursion in (P4.2). This completes the proof. □

A.2.7 Proof of corollary 1

Proof of corollary 1. For the cases where $\exists k \in \mathbb{Z}_{[0,K]}$ such that $t = \tau^k$, use the definition (21) along with the smoothed density in (11) to obtain,

$$\widehat{Z}_t = \mathbb{E}\left[Z^\star|z^T;\widehat{\Theta}^c\right] = \int z^\star \delta(z^\star - c^k)dz^\star = c^k \qquad \text{(C1.1)}$$

This follows from the properties of the Dirac delta function.

For the remaining cases, the key fact to notice is that if $\underline{x} = -\infty$ and $\bar{x} = +\infty$ then $\underline{Z}^k \equiv \mathbb{1}\{k = 0\}x_0 + \mathbb{1}\{k \geqslant 1\}c^k - \bar{x} = -\infty$ and $\bar{Z}^k \equiv \mathbb{1}\{k = 0\}x_0 + \mathbb{1}\{k \geqslant 1\}c^k - \underline{x} = \infty$. If the boundaries of the inaction region diverge, then for a given $k \in \mathbb{Z}_{[0,K-1]}$ and $b \in [1, \Delta^k - 1]$ or $k = K$ and $b \in [1, \Delta^K]$ it holds that $\beta_b^k(x) = 1$, $\chi_b^k(x) = 1$ and $\iota_b^k(x) = 1$ $\forall x \in \mathbb{R}$. To formally verify this consider first the recursion $\beta_b^k(x)$ as defined in (13) and verify this by induction. Fix k equal to some value in $\mathbb{Z}_{[0,K]}$. Consider the base case $b = 1$. By definition, $\beta_1^k(x) = 1, \forall x \in \mathbb{R}$ so the base case is trivially satisfied. For the induction step, suppose $\beta_b^k(x) = 1, \forall x$ for some $b > 1$then using the definition in (13),

$$\beta_{b+1}^k(x) = \int_{\underline{Z}^k}^{\bar{Z}^k} \frac{1}{\tilde{\sigma}_b}\,\phi\left(\frac{y - \tilde{\mu}_b^k(x)}{\tilde{\sigma}_b}\right)\, \beta_b^k\,(y)\,dy = \int_{-\infty}^{\infty} \frac{1}{\tilde{\sigma}_b}\,\phi\left(\frac{y - \tilde{\mu}_b^k(x)}{\tilde{\sigma}_b}\right)\,dy = 1 \qquad \text{(C1.2)}$$

where the last equality follows from the fact that it is an integral of a normal density from $-\infty$ to ∞. Similarly, for a given k equal to some value in $\mathbb{Z}_{[0,K]}$, it is the case that $\chi_{\Delta^k - 1}^k(x) = 1$ by definition. For the induction step, suppose that $\chi_{b+1}^k(x) = 1$ for some $b < \Delta^k - 2$. Then using the definition in (14) and similarly to above,

$$\chi_b^k(x) = \int_{\underline{Z}^k}^{\bar{Z}^k} \frac{1}{\ddot{\sigma}_{b+1}^k}\,\phi\left(\frac{y - \ddot{\mu}_{b+1}^k(x)}{\ddot{\sigma}_{b+1}^k}\right)\, \chi_{b+1}^k\,(y)\,dy = \int_{-\infty}^{\infty} \frac{1}{\ddot{\sigma}_{b+1}^k}\,\phi\left(\frac{y - \ddot{\mu}_{b+1}^k(x)}{\ddot{\sigma}_{b+1}^k}\right)dy = 1 \quad \text{(C1.3)}$$

where again the last equality follows from the fact that it is an integral of a normal density from $-\infty$ to ∞. The proof for $\iota_b^k(x)$ is analogous to that of $\chi_b^k(x)$ in (C1.3).

For t such that $\exists k \in \mathbb{Z}_{[0,K-1]}$ such that $t \in (\tau^k, \tau^{k+1})$ it is the case that,

$$\widehat{Z}_t = \mathbb{E}\left[Z^\star | z^T; \widehat{\Theta}^c\right] = \int z^\star \frac{1}{\breve{\sigma}_b^k} \phi\left(\frac{z^\star - \breve{\mu}_b^k}{\breve{\sigma}_b^k}\right) \beta_b^k(z^\star)\, \chi_b^k(z^\star)\, \mathbb{1}\{z^\star \in \mathcal{I}^k\} dz^\star$$
$$= \int z^\star \frac{1}{\breve{\sigma}_b^k} \phi\left(\frac{z^\star - \breve{\mu}_b^k}{\breve{\sigma}_b^k}\right) dz^\star$$
$$= \breve{\mu}_b^k \tag{C1.4}$$

For t such that $t \in (\tau^k, T]$ it is the case that,

$$\widehat{Z}_t = \mathbb{E}\left[Z^\star | z^T; \widehat{\Theta}^c\right] = \int z^\star \frac{1}{\sigma_b} \phi\left(\frac{z^\star - \mu_b^K}{\sigma_b}\right) \beta_b^K(z^\star)\, \iota_b^K(z^\star)\, \mathbb{1}\{z^\star \in \mathcal{I}^K\} dz^\star$$
$$= \int z^\star \frac{1}{\sigma_b} \phi\left(\frac{z^\star - \mu_b^K}{\sigma_b}\right) dz^\star$$
$$= \breve{\mu}_b^k \tag{C1.5}$$

where in both (C1.4) and (C1.5) the first equality just uses the definition of smoothed estimates in (21) with the smoothed pdf in (12) and (18), respectively. The second equality holds since, under the special case with $\underline{x} = -\infty$ and $\bar{x} = \infty$, it is the case that $\beta_b^k(x) = 1$, $\chi_b^k(x) = 1$ and $\iota_b^k(x) = 1\ \forall x \in \mathbb{R}$ and $\mathcal{I}^k = \mathbb{R}$ for any $k \in \mathbb{Z}_{[0,K]}$.

This completes the proof. □

B Computational details for estimation and smoothing

This appendix contains computational details for parameter estimation and for the computation of the smoothed density estimates as described in section 3.All the procedures here described are implemented in MATLAB R2015b.[49]

B.1 Details for the parameter estimation

Parameter estimation is done in two stages as described in section 3. In this section, we briefly describe the algorithms used to obtain parameter estimates for common parameters in the first stage and to obtain the estimates of initial price gaps in the second stage.

B.1.1 Algorithm for estimation of common parameters

In general, the vector of simulated price changes $\Delta p^s_{j|t>\tau^1}$ depends on the vector of common parameters θ_j and the vector of primitive shocks used to generate that panel of simulated data $\xi^s = \{\{\varepsilon_{i,j,t}, \nu_{i,j,t}\}^{T_{i,j}}_{\tau^0_{i,j}}\}^{\mathrm{N}_j}_{i=1}$.[50] More precisely, one should write $\Delta p^s_{j|t>\tau^1} = f(\xi^s, \theta_j)$ where the function $f(\cdot)$ is implicitly defined by the state-space representation in (5) to (7). Moreover, for a sequence of S vectors of shocks $\Xi = [\xi^1, \ldots, \xi^S]$, let $G(\Delta p_{j|t>\tau^1}, \theta_j, \Xi)$ denote the value of (9) where $\Delta p_{j|t>\tau^1}$ be the vector of price changes excluding the first observed in the data.

Algorithm 1 The algorithm for minimisation of $G(\Delta p_{j|t>\tau^1}, \theta_j, \Xi)$ is as follows:

1. Draw 50 vectors of shocks conform with the data template. Let Ξ^0 denote that vector.
2. Choose an initial value for the vector of parameters, say $\theta^{(0)}_j$.
3. Use a global search algorithm to search for the minimiser of $G(\Delta p_{j|t>\tau^1}, \theta_j, \Xi^0)$.[51]
4. The search is subject to the restrictions: $\underline{x}_j \leqslant 0$, $\bar{x}_j \geqslant 0$, $\sigma_{\varepsilon,j} \geqslant 0$ and $\lambda_j \in [0, 1]$.

Some points about algorithm 1 above are worth emphasising. First, in step 3 we use global search methods since in preliminary simulations gradient based methods failed to converge in many instances. Second, the simulated data is generated according to the same data template observed in the actual data in accordance with the general principle in simulated based estimation of treating real and simulated data as similarly as possible. Third, the vectors of primitive shocks in Ξ^0 are drawn only *once* at the beginning of the algorithm and kept fixed when searching for a minimum. Otherwise, the algorithm would not numerically converge and the asymptotic statistical properties would no longer be valid.[52] Fourth, for each product we

[49] Codes used for parameter estimation and computation of smoothed estimates are available from the author upon request.

[50] Note that the drawings of the shocks are done such that in simulated data the number of quote lines and their respective starting and ending dates exactly match those that are observed in actual data.

[51] We use the algorithm *patternsearch* in MATLAB R2015b with the default options

[52] See, for instance, p. 29 in Gouriéroux and Monfort (1996).

run steps 2 to 4 twice starting two different initial conditions in step 2.[53] In case the results differ, we choose the value of parameters that results in the smallest objective function.

B.1.2 Algorithm for estimation of initial price gaps

For the estimation of initial conditions of quote-lines of a given product, the minimisation problem in (10) has to be solved once for each quote-line that has at least one non-zero price change. Given the large number of such quote-lines performing separate global search methods for each quote line individually as for the estimation of common parameters (algorithm 1) would be infeasible. Instead, we perform the minimisation on a grid of possible values of $x_{i,j,0}$. More precisely, to estimate the initial conditions for all the quote-lines of a given product we implement the following algorithm:

Algorithm 2 The algorithm for estimating initial conditions is as follows:

1. Determine the number of panels to be simulated as $S = \lceil 10^4/N_j \rceil$.[54]
2. Generate S vectors of primitive shocks conform with the data template, $\Xi = [\xi^1, \ldots, \xi^S]$.
3. Create a grid $\mathcal{G} = [x^{(1)}_{i,j,0}, \ldots, x^{(50)}_{i,j,0}]$, where $x^{(k)}_{i,j,0} = \hat{\underline{x}}_j + \frac{k}{(50+1)}(\hat{\bar{x}}_j - \hat{\underline{x}}_j)$.
4. Set $\theta = \hat{\theta}$ and $x_{i,j,0} = x^{(k)}_{i,j,0}, \forall i$ and use (5) to (7) and Ξ to generate S panels of data.
5. For a given collection of panels compute $f(x^{(k)}_{i,j,0}) = (S\, N_j)^{-1} \sum_{s=1}^{S} \sum_{i=1}^{N_j} h(\Delta p^s_{i,j}(x^{(k)}_{i,j,0}, \hat{\theta}_j))$
6. Repeat steps 4 and 5 for each $x^{(k)}_{i,j,0} \in \mathcal{G}$ and store $\mathcal{F} = \{f(x^{(k)}_{i,j,0})\}_{k=1}^{50}$
7. Create a new grid $\tilde{\mathcal{G}} = [\tilde{x}^{(1)}_{i,j,0}, \ldots, \tilde{x}^{(50{,}000)}_{i,j,0}]$ where $\tilde{x}^{(i)}_{i,j,0} = x^{(1)}_{i,j,0} + \frac{(i-1)}{(49{,}999)}(x^{(50)}_{i,j,0} - x^{(1)}_{i,j,0})$
8. For each $\tilde{x}^{(k)}_{i,j,0} \in \tilde{\mathcal{G}}$ use a cubic spline on the values in $\mathcal{F}$ to approximate $f(\tilde{x}^{(i)}_{i,j,0})$.[55]
9. For each $\tilde{x}^{(k)}_{i,j,0} \in \tilde{\mathcal{G}}$ compute $\tilde{H}(\Delta p_{i,j}, \tilde{x}^{(k)}_{i,j,0}, \hat{\theta}_j) = \left\| \left(h(\Delta p_{i,j}) - f(\tilde{x}^{(k)}_{i,j,0}) \right) \oslash h(\Delta p_{i,j}) \right\|^2$
10. For a given quote-line with at least one price change take $\hat{x}_{i,j,0} = \arg \min\limits_{a \in \tilde{\mathcal{G}}} \tilde{H}(\Delta p_{i,j}, a, \hat{\theta}_j)$
11. Repeat steps 9 and 10 for each quote-line with at least one non-zero price change.
12. For quote-lines without price changes, set $\hat{x}_{i,j,0}$ equal to the average of values in step 11.

Some points about algorithm 2 are worth emphasising. First, in step 9 the deviations of data moments from their simulated counterparts are expressed as percentage deviations of the data

[53] The first set of initial conditions is designed to be an educated guess for a model that is closer to a pure menu cost model. In that case, we set the initial values for $-\underline{x}_j$ and $-\bar{x}_j$ to be equal to the average values of positive and negative log price changes, respectively, whereas initial value for λ_j is equal to 25% of the frequency of price changes in the data. The second initial condition is designed to be an educated guess for a model that is closer to a pure Calvo model. In that case, we set $-\underline{x}_j$ and $-\bar{x}_j$ to be equal to the 95th and 5th percentiles of the distribution of log price changes, respectively, and λ_j equal to 75% of the frequency of price changes observed in the data.

[54] This ensures the simulator is based on at least 10 thousand individual price trajectories.

[55] We use the function *interp1* in MATLAB R2015b with the option 'spline'.

moments and that is necessary with equally weighted moments to ensure that one moment condition does not receive a disproportional weight simply due to differences in scale. Second, once the grid for the approximation of the moment is constructed (steps 1 to 8) it can be used as the moment simulator for other quote-lines, in practice, this implies that only steps 9 and 10 need to be repeated at the quote-line each speeds up the calculations. Third, the initial grid $\mathcal{F}$ could be equivalently generated by simply generating 10,000 separate individual price trajectories all starting from a given initial condition and computing average over those. The construction in steps 1 to 6 simply takes advantage of some functions used to generate simulated data for the estimation of common parameters.

B.2 Details on the computation of smoothed estimates

From a purely computational perspective, once a vector of parameters is estimated the key challenge to compute smoothed estimates is to be able to numerically evaluate integrals sufficiently fast so that such estimates can be computed for the millions of observations in micro price data. In order to do that, we numerically approximate integrals using *Gauss-Legendre quadrature* methods.[56] As in the main text, and without loss of generality, henceforth we consider an arbitrary quote-line with $K \geqslant 0$ non-zero price changes and fix k equal to some value $\mathbb{Z}_{[0,K]}$. For that quote-line, τ^k denotes the period on which the k-th non-zero price change is observed and τ^0 denotes the initial period. In addition, let n denote the number of Gauss-Legendre nodes used in the approximation, let $\mathbf{z}$ be a $n \times 1$ vector of Gauss-Legendre nodes in the interval $[\underline{Z}^k, \bar{Z}^k]$ and $\boldsymbol{\omega}$ be the associated $n \times 1$ vector of Gauss-Legendre weights. Finally, let z_i and ω_i denote the i-th elements of $\mathbf{z}$ and $\boldsymbol{\omega}$, respectively.[57]

Matrix notation Specifically to describe the algorithms in this subsection, we use: T as a superscript to denote matrix transposition; $a_{*,j}$ to denote the j-th column of the matrix $\mathbf{A}$; $a_{i,*}$ to denote the i-th row of $\mathbf{A}$; $\mathbf{I}_n$ denotes the identity matrix of order n; $\mathbb{1}_{m\times n}$ denotes an $m \times n$ matrix of ones; $\otimes$ denotes the Kronecker product; $\odot$ denotes the Hadamard product; $\oslash$ denotes the Hadamard division and the exponent $^{\circ -1}$ the Hadamard inverse. Finally, for a given function $f : \mathbb{R} \to \mathbb{R}$ and a matrix $\mathbb{A} \in \mathbb{R}^{m\times n}$, the notation $B = f \circ (\mathbf{A})$ is equivalent to $b_{i,j} = f(a_{i,j})$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Any other notation is as defined in the main text.

B.2.1 Smoothed estimates based on the the pdf in Proposition 3

Consider first the case where $k < K$. For $i = 1, \ldots, n$ and $j = 1, \ldots, \Delta^k - 1$, define:

[56] See, for example, Judd (1998, section 7.2).

[57] To compute the Gauss-Legendre nodes and associated weights on an arbitrary interval $[a, b]$, we use the *lgwt* function provided by Greg von Winckel on File Exchange and available for download here. Here we describe the computations for a general number of Gauss-Legendre notes n. In practice, in any integral numerical evaluation we use 50 Gauss-Legendre nodes.

$$a_{i,j} \equiv \frac{1}{\breve{\sigma}_j^k}\,\phi\left(\frac{z_i - \breve{\mu}_j^k}{\breve{\sigma}_j^k}\right) \tag{B.2.1.1}$$

$$b_{i,j} \equiv \mathbb{1}\{j>1\}\left[\sum\nolimits_{l=1}^{n} \omega_l\,\frac{1}{\tilde{\sigma}_{j-1}}\phi\left(\frac{z_l - \tilde{\mu}_{j-1}^k(z_i)}{\tilde{\sigma}_{j-1}}\right) b_{l,j-1}\right] + \mathbb{1}\{j=1\} \tag{B.2.1.2}$$

$$c_{i,j} \equiv \mathbb{1}\{j<\Delta^k-1\}\left[\sum\nolimits_{l=1}^{n} \omega_l\,\frac{1}{\ddot{\sigma}_{j+1}}\phi\left(\frac{z_l - \ddot{\mu}_{j+1}^k(z_i)}{\ddot{\sigma}_{j+1}}\right) c_{l,j+1}\right] + \mathbb{1}\{j=\Delta^k-1\} \tag{B.2.1.3}$$

For a given $j \in \mathbb{Z}_{[1,\Delta^k-1]}$, the smoothed estimates as defined in (21) can be obtained as:

$$\widehat{Z}^{\star}_{\tau^k+j} = \frac{\sum\limits_{l=1}^{n} \omega_l\, a_{l,j}\, b_{l,j}\, c_{l,j}\, z_l}{\sum\limits_{l=1}^{n} \omega_l\, a_{l,j}\, b_{l,j}\, c_{l,j}} \tag{B.2.1.4}$$

Matrix form For computational efficiency it is preferable to implement the calculation in (B.2.1.4) in matrix form. For that purpose, first define $\breve{\mu}_*^k = [\breve{\mu}_1^k, \ldots, \breve{\mu}_{\Delta^k-1}^k]$ and $\breve{\sigma}_*^k = [\breve{\sigma}_1^k, \ldots, \breve{\sigma}_{\Delta^k-1}^k]$. Let $\mathbf{A}$ be a $n \times (\Delta^k - 1)$ matrix given by,

$$\mathbf{A} = \left[\mathbb{1}_{n\times 1} \otimes (\breve{\sigma}_*^k)^{\circ-1}\right] \odot \left[\phi \circ \left(\, (\mathbb{1}_{1\times(\Delta^k-1)} \otimes \mathbf{z} - \mathbb{1}_{n\times 1} \otimes \breve{\mu}_*^k) \oslash (\mathbb{1}_{n\times 1} \otimes \breve{\sigma}_*^k)\,\right)\right] \tag{B.2.1.5}$$

Note that the (i,j) element of $\mathbf{A}$ in (B.2.1.5) is equal to (B.2.1.1). Moreover, let $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{n\times(\Delta^k-1)}$ be such that $b_{*,1} = \mathbb{1}_{n\times 1}$ and $c_{*,\Delta^k-1} = \mathbb{1}_{n\times 1}$ and the remaining columns are defined recursively according to:

$$b_{*,j} = \left[\,\mathbf{I}_n \otimes \omega^T\,\right]\left[\frac{1}{\tilde{\sigma}_{j-1}} \times \phi \circ \left(\frac{1}{\tilde{\sigma}_{j-1}}\left(\mathbb{1}_{n\times 1} \otimes \mathbf{z} - \tilde{\mu}_j^k \circ (\mathbf{z} \otimes \mathbb{1}_{n\times 1})\right)\right) \odot (\mathbb{1}_{n\times 1} \otimes b_{*,j-1})\right] \tag{B.2.1.6}$$

$$c_{*,j} = \left[\,\mathbf{I}_n \otimes \omega^T\,\right]\left[\frac{1}{\ddot{\sigma}_{j+1}^k} \times \phi \circ \left(\frac{1}{\ddot{\sigma}_{j+1}^k}\left(\mathbb{1}_{n\times 1} \otimes \mathbf{z} - \ddot{\mu}_j^k \circ (\mathbf{z} \otimes \mathbb{1}_{n\times 1})\right)\right) \odot (\mathbb{1}_{n\times 1} \otimes c_{*,j+1})\right] \tag{B.2.1.7}$$

The (i,j) element of $\mathbf{B}$ and $\mathbf{C}$ are equal to those defined in (B.2.1.2) and (B.2.1.3). Finally, the smoothed estimates can be obtained as:

$$\widehat{\mathbf{Z}}^k = \left[\omega^T \left((\mathbb{1}_{1\times(\Delta^k-1)} \otimes \mathbf{z}) \odot \mathbf{A} \odot \mathbf{B} \odot \mathbf{C}\right)\right] \oslash \left[\omega^T \left(\mathbf{A} \odot \mathbf{B} \odot \mathbf{C}\right)\right] \tag{B.2.1.8}$$

where $\widehat{\mathbf{Z}}^k \in \mathbb{R}^{1\times(\Delta^k-1)}$ and the j-th element of $\widehat{\mathbf{Z}}^k$ is equal to $\widehat{Z}^\star_{\tau^k+j}$ as given in (B.2.1.4).

B.2.2 Smoothed estimates based on the the pdf in Proposition 4

For $k = K$ the smoothed pdf is given by (18). Numerically, the computation of the smoothed estimates is similar to that in the previous section. For $i = 1, \ldots, n$ and $j = 1, \ldots, \Delta^K$, define:

$$d_{i,j} \equiv \frac{1}{\sigma_j}\,\phi\left(\frac{z_i - \mu_j^k}{\sigma_j}\right) \tag{B.2.2.1}$$

$$e_{i,j} \equiv \mathbb{1}\{j < \Delta^K\}\left[\sum\nolimits_{l=1}^{n} \omega_l\,\frac{1}{\sigma_\varepsilon}\phi\left(\frac{z_l - (\mu + z_i)}{\sigma_\varepsilon}\right) e_{l,j+1}\right] + \mathbb{1}\{j = \Delta^K\} \tag{B.2.2.2}$$

For a given $j \in \mathbb{Z}_{(\tau^K,T]}$, the smoothed estimates as defined in (21) can be obtained as:

$$\widehat{Z}^\star_{\tau^k+j} = \frac{\sum\limits_{l=1}^{n} \omega_l\, d_{l,j}\, b_{l,j}\, e_{l,j}\, z_l}{\sum\limits_{l=1}^{n} \omega_l\, d_{l,j}\, b_{l,j}\, e_{l,j}} \tag{B.2.2.3}$$

Matrix form Again, for computational efficiency we implement the computation of (B.2.2.3) in matrix form. First, define $\mu_* = [\mu_1, \ldots, \mu_{\Delta^K}]$ and $\sigma_* = [\sigma_1, \ldots, \sigma_{\Delta^K}]$. Let $\mathbf{D}$ be a $n \times \Delta^K$ matrix given by,

$$\mathbf{D} = \left[\mathbb{1}_{n\times 1} \otimes (\sigma_*)^{\circ -1}\right] \odot \left[\phi \circ \left(\,(\mathbb{1}_{1\times\Delta^K} \otimes \mathbf{z} - \mathbb{1}_{n\times 1} \otimes \mu_*) \oslash (\mathbb{1}_{n\times 1} \otimes \sigma_*)\,\right)\right] \tag{B.2.2.4}$$

Note that the (i, j) element of $\mathbf{D}$ is equal to (B.2.2.1). Let $\tilde{\mathbf{B}}, \mathbf{E} \in \mathbb{R}^{n\times\Delta^K}$ be such that $\tilde{b}_{*,1} = \mathbb{1}_{n\times 1}$ and $e_{*,\Delta^K} = \mathbb{1}_{n\times 1}$. The remaining columns of $\tilde{\mathbf{B}}$ are defined recursively according to (B.2.1.6) and the remaining columns of $\mathbf{E}$ according to:

$$e_{*,j} = \left[\,\mathbf{I}_n \otimes \omega^T\,\right]\left[\frac{1}{\sigma_\varepsilon} \times \phi \circ \left(\frac{1}{\ddot{\sigma}^k_{j+1}}\left(\mathbb{1}_{n\times 1} \otimes \mathbf{z} - (\mu\mathbb{1}_{n\times 1} + \mathbf{z}) \otimes \mathbb{1}_{n\times 1}\right)\right) \odot \left(\mathbb{1}_{n\times 1} \otimes e_{*,j+1}\right)\right] \tag{B.2.2.5}$$

Finally, the smoothed estimated after the last non-zero price change are computed from:

$$\widehat{\mathbf{Z}}^K = \left[\omega^T \left((\mathbb{1}_{1\times(\Delta^k-1)} \otimes \mathbf{z}) \odot \mathbf{D} \odot \tilde{\mathbf{B}} \odot \mathbf{E}\right)\right] \oslash \left[\omega^T \left(\mathbf{D} \odot \tilde{\mathbf{B}} \odot \mathbf{E}\right)\right] \tag{B.2.2.6}$$

where $\widehat{\mathbf{Z}}^K \in \mathbb{R}^{1\times\Delta^K}$ and the j-th element of $\widehat{\mathbf{Z}}^K$ is equal to $\widehat{Z}^{\star}_{\tau^K+j}$ as given in (B.2.2.3).

C Frictionless inflation in the basic new Keynesian model

The underlying model and all the notation are identical to the basic new Keynesian model in Galí (2008, chapter 3). We focus only on the key equations to derive: (i) the relation between frictionless inflation and the output gap; (ii) the difference in the responses of inflation and frictionless inflation to a monetary policy shock.

C.1 Relationship between the inflation, frictionless inflation and the output gap

Under monopolistic competition and a demand function arising from a CES aggregator with elasticity of substitution ε, frictionless prices satisfy:

$$P_t^\star = \mathcal{M}\,\psi_{t|t} \tag{C.1.1}$$

where $\psi_{t|t}$ are the nominal marginal costs of a firm changing prices at time t and $\mathcal{M} \equiv \varepsilon/(\varepsilon-1)$ is a constant markup that monopolist would charge at every time period in the absence of constraints on the frequency of price adjustment, also referred to as the desired or frictionless markup. Dividing both sides of (C.1.1) by P_t and taking logs yields,

$$p_t^\star - p_t = mc_t - mc \tag{C.1.2}$$

where $mc = -\log(\mathcal{M})$ is the steady state value of marginal cost and mc_t is the log of the economy's average real marginal cost. Since the log of deviation of real marginal cost from steady state is proportional to the log deviation of output from its flexible price counterpart, we use equation (20) in Galí (2008, p. 48) to obtain,

$$p_t^\star - p_t = \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)\tilde{y}_t \tag{C.1.3}$$

where $\tilde{y}_t$ is the *output gap*, defined as the log deviation of output from its flexible part counterpart, σ is the elasticity of intertemporal substitution, φ is the Frisch elasticity of labor supply and $1-\alpha \in [0,1]$ is the exponent of labor in the production function. Lagging (C.1.3) by one period and subtracting from (C.1.3),

$$\pi_t^\star - \pi_t = C\,\Delta\tilde{y}_t \tag{C.1.4}$$

where $C \equiv \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right) > 0$. Taking the covariance between $\pi_t - \pi_t^\star$ and $\Delta\tilde{y}_t$ gives equation (26), that is,

$$\frac{\mathrm{Cov}(\pi_t - \pi_t^\star, \tilde{y}_t)}{\mathrm{Var}(\tilde{y}_t)} = -C < 0 \tag{C.1.5}$$

Tests for this prediction in the UK data are presented in table 3.

C.2 Impulse responses to a monetary policy shock

We consider the responses of inflation and frictionless inflation under an interest rate rule as in Galí (2008, section 3.4.1). The stochastic component in the interest rate is v_t and it is assumed to follow an AR(1) process, that is, $v_t = \rho_v v_{t-1} + \varepsilon_t^m$ where $\rho_v \in [0, 1)$ and ε_t^m is the monetary policy shock. From (C.1.4) and any given horizon $h \geqslant 0$ we have that,

$$\frac{\partial \pi^\star_{t+h}}{\partial \varepsilon_t^v} - \frac{\partial \pi_{t+h}}{\partial \varepsilon_t^m} = C \, \frac{\partial \Delta \tilde{y}_{t+h}}{\partial \varepsilon_t^m} \tag{C.2.6}$$

So the relationship between the impulse responses of frictionless inflation and inflation can be inferred from the sign of the impulse response of the changes in the output gap. Using the method of undetermined coefficients, the solution for the output gap is given by,

$$\tilde{y}_{t+h} = -(1 - \beta \rho_v) \, \Lambda_v \, v_{t+h} \tag{C.2.7}$$

where β is the representative household discount factor and Λ_v is a convolution of structural parameters that and takes only positive values.[58] Finally, in terms of impulse responses it is the case that,

$$\frac{\partial \Delta \tilde{y}_{t+h}}{\partial \varepsilon_t^m} = -(1 - \beta \rho_v) \, \Lambda_v \left(\frac{\partial v_{t+h}}{\partial \varepsilon_t^m} - \frac{\partial v_{t+h-1}}{\partial \varepsilon_t^m} \right) \tag{C.2.8}$$

Note that for the case $h = 0$, the last term in brackets in (C.2.9) is equal to one since $\partial v_{t-1} / \partial \varepsilon_t^m = 0$. Since the term $(1 - \beta \rho_v) \Lambda_v$ is positive, it follows that the expression above is *negative* and, hence, $\partial \pi^\star_t / \partial \varepsilon_t^v < \partial \pi_t / \partial \varepsilon_t^m$. Moreover, since the solution for inflation is given by $\pi_t = -\kappa \, \Lambda_v \, v_t$ where $\kappa > 0$ is the slope of the new Keynesian Phillips curve and, hence, $\partial \pi_t / \partial \varepsilon_t^m < 0$. Therefore, (28) holds. Finally, for any $h > 0$ expression (C.2.9) simplifies to,

$$\frac{\partial \Delta \tilde{y}_{t+h}}{\partial \varepsilon_t^m} = - \underbrace{(1 - \beta \rho_v) \, \Lambda_v \rho_v^{h-1}}_{>0} \, \underbrace{(\rho_v - 1)}_{<0} \, > 0 \tag{C.2.9}$$

Therefore, $\partial \pi^\star_{t+h} / \partial \varepsilon_t^v > \partial \pi_{t+h} / \partial \varepsilon_t^m$ as stated in (29). As an endnote, notice that the sign of $\pi_{t+h} / \partial \varepsilon_t^m$ cannot be determined unambiguously. More precisely using the solutions for π_t and $\Delta \tilde{y}_t$ and the relationship in (C.1.4),

$$\frac{\partial \pi^\star_{t+h}}{\partial \varepsilon_t^v} = \kappa \Lambda_v \rho_v^{h-1} \left[\frac{(1 - \beta \rho_v)(1 - \rho_v)}{\lambda} - \rho_v \right] \tag{C.2.10}$$

[58] For values of structural parameters that ensure equilibrium uniqueness which is maintained assumption, see Galí (2008, equation 27).

where λ is a positive constant that is a convolution of structural parameters. The sign of the last term in square brackets will depend on the specific calibration of structural parameters.